The Ricci Flow for Nilmanifolds

Tracy L. Payne

December 11, 2008

Abstract

We consider the Ricci flow for simply connected nilmanifolds, which translates to a Ricci flow on the space of nilpotent metric Lie algebras. We consider the evolution of the inner product with respect to time and the evolution of structure constants with respect to time, as well as the evolution of these quantities modulo rescaling. We set up systems of O.D.E.'s for some of these flows and describe their qualitative properties. We also present some explicit solutions for the evolution of soliton metrics under the Ricci flow.

1 Introduction

The Ricci flow, defined by R. Hamilton ([Ham82]), is an important tool for understanding the topology and geometry of three-manifolds. It is key in Perelman's work ([Pera], [Perc], [Perb]), and has useful applications in other areas of geometry as well. Due to the difficulty of solving the partial differential equations involved, very few explicit examples of Ricci flow solutions are known.

For a homogeneous manifold, the Ricci flow can be presented as a set of O.D.E.'s rather than P.D.E.'s. In many cases, it is possible to solve these systems exactly or to make estimates that allow a description of the qualitative behavior of the system. The Ricci flows for the universal covers of compact homogeneous spaces of dimension three have been analyzed in [IJ92] and [KM01]. Solutions for certain metrics on the universal covers of compact homogeneous four-manifolds are described in [IJL06].

These examples provide insight into the behavior of the Ricci flow in general, exhibiting many of the phenomena of interest in variable curvature cases. If a manifold M admits a homogeneous metric, the set of homogeneous metrics on M is invariant under the Ricci flow on the space of all metrics on M. One hopes that this invariant set is an attractor for the Ricci flow. If that is true, a description of the Ricci flow for homogeneous metrics on M would yield an understanding of the long-term behavior of metrics in a large subset of the set of all metrics on M.

In this work, we continue the study of the Ricci flow for homogeneous spaces initiated in [IJ92], [KM01], and [IJL06]. We analyze the Ricci flow for the class of homogeneous spaces consisting of simply connected nilmanifolds of arbitrary dimension. A *nilmanifold* is a Riemannian manifold with universal cover (N, g), where N is a simply-connected nilpotent Lie group N endowed with a left-invariant metric g. Although no Einstein metrics exist on a nilpotent Lie group N unless N is abelian (Corollary 2, [Jen69]), many nilpotent groups admit soliton metrics. A *soliton metric* is a metric g that evolves under the Ricci flow by diffeomorphisms and rescaling; that is, $g_t = t \cdot \eta_t^* g_0$, where η_t is a one-parameter family of diffeomorphisms. If a nilpotent Lie group does admit a soliton metric, then it is unique up to scaling ([Lau01]). However, not all nilpotent Lie groups admit soliton metrics. Using geometric invariant theory, it can be shown that under the Ricci flow, any left-invariant metric g on a nilpotent Lie group N approaches, modulo rescaling, a unique soliton metric g' on a nilpotent Lie group N', and if N admits a soliton metric, that limiting nilmanifold is the soliton metric on N. ([Jab]).

The Ricci flow for three- and four-dimensional nilmanifolds is fairly well understood. In dimension three, there is a single simply connected nonabelian nilpotent Lie group, the Heisenberg group H_3. It is shown in [IJ92] that under the Ricci flow, any initial left-invariant metric g_0 on H_3 collapses to a flat metric on $\mathbb{R}^2$. In dimension four, there is a single simply connected nilpotent Lie group that is not a product of lower-dimensional nilpotent Lie groups, the filiform group L_4. It was shown in [IJL06] that under the Ricci flow, any initial metric g_0 that is diagonal with respect to a special basis collapses. As time goes to infinity, any initial metric on H_3 and the special metrics on L_4 can be viewed in the appropriate framework as asymptotically projectively approaching the unique soliton metric on H_3 or L_4, respectively (See [Lot07], [GIK06]).

Now we summarize the main results in this paper. Following this section, in Section 2 we establish the necessary background and preliminaries involving nilmanifold geometry and the Ricci flow. The Ricci flow on the space of left-invariant metrics on a simply connected Lie group G can be converted to a flow on the space of inner products on the corresponding Lie algebra $\mathfrak{g}$. The flows for all individual metric Lie algebras can be combined to define a Ricci flow on the space of metric Lie algebras, and this flow projects to a projectivized Ricci flow on the space of all volume-normalized metric Lie algebras. We define what it means for for a metric Lie algebra to projectively approach another metric Lie algebra, and what it means for a metric Lie algebra to collapse under the Ricci flow. We define a Lie bracket flow for a Lie algebra that describes how the Lie bracket relative to a moving orthonormal basis changes under the Ricci flow, and we define a projectivization of this flow.

In Theorem 3.1 of Section 3, we set up systems of O.D.E.'s for the Ricci flow and the Lie bracket flow for a single nilpotent Lie algebra $(\mathfrak{n}, \mathsf{Q})$. Theorem 3.6 gives

O.D.E.'s for the projectivized Lie bracket flow. Proposition 3.9 describes invariant quantities for the Ricci flow for nilpotent metric Lie algebras.

In Section 4, we find some explicit solutions for soliton trajectories for the Ricci flow for nilpotent metric Lie algebras; these are presented in Theorem 4.2. The theorem only gives solutions for nilpotent metric Lie algebras admitting a special kind of basis. We describe some broad conditions under which these bases exist.

Finally, in Section 5, we give some examples. We consider the Lie bracket flow for Heisenberg Lie algebras and a Lie algebra that does not admit a soliton metric.

The author is grateful to Jim Isenberg for his suggestion of studying the Ricci flow for nilmanifolds. She thanks him, Jorge Lauret, Peng Lu and Dave Glickenstein for helpful discussions. This work was supported by NSF ADVANCE grant #SBE-0620073.

2 Preliminaries

2.1 Structure of metric nilpotent Lie algebras

Suppose that $(\mathfrak{g}, \mathsf{Q})$ is an n-dimensional metric Lie algebra with basis $\mathcal{B} = \{\boldsymbol{x}_i\}_{i=1}^n$. The set

$$\Lambda_{\mathcal{B}} = \{(j,k,l) \,|\, \alpha_{jk}^l \neq 0, 1 \leq j < k \leq n, 1 \leq l \leq n\}$$

indexes the set of nonzero structure constants α_{jk}^l for $\mathfrak{g}$ relative to $\mathcal{B}$ without repetitions due to skew-symmetry.

Suppose that $\mathfrak{g}$ is nonabelian, so $\Lambda_{\mathcal{B}}$ is nonempty. Let $\{\boldsymbol{e}_i\}_{i=1}^n$ be the standard orthonormal basis for $\mathbb{R}^n$. For $1 \leq j,k,l \leq n$, define the $1 \times n$ row vector $\boldsymbol{y}_{jk}^l$ to be $\boldsymbol{e}_j^T + \boldsymbol{e}_k^T - \boldsymbol{e}_l^T$. We call a vector $\boldsymbol{y}_{jk}^l$, where $(j,k,l) \in \Lambda_{\mathcal{B}}$, a *root vector* for $(\mathfrak{g}, \mathsf{Q})$ relative to the basis $\mathcal{B}$. Let $\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_m$ (where $m = |\Lambda_{\mathcal{B}}|$) be an enumeration of the root vectors $\boldsymbol{y}_{jk}^l$ for $(\mathfrak{g}, \mathsf{Q})$ relative to $\mathcal{B}$, in the dictionary order on the integer triples (j,k,l). Define the *root matrix* Y for $(\mathfrak{g}, \mathsf{Q})$ relative to $\mathcal{B}$ to be the $m \times n$ matrix whose rows are the root vectors $\boldsymbol{y}_1, \boldsymbol{y}_2, \cdots, \boldsymbol{y}_m$. When $\mathfrak{g}$ is nonabelian, the *Gram matrix* U for $(\mathfrak{g}, \mathsf{Q})$ relative to $\mathcal{B}$ is defined to be the $m \times m$ matrix $U = YY^T$.

Suppose that the basis $\mathcal{B}$ is orthonormal, so that the inner product Q can be written as $\mathsf{Q} = \sum_{i=1}^n dx^i \otimes dx^i$, where dx^i is dual to $\boldsymbol{x}_i$, for $i = 1, \ldots, n$. The structure constants for $\mathfrak{g}$ relative to $\mathcal{B}$ are given by

$$\alpha_{jk}^l = \mathsf{Q}([\boldsymbol{x}_j, \boldsymbol{x}_k], \boldsymbol{x}_l) \tag{1}$$

for $1 \leq j,k,l \leq n$. Letting $\mathsf{Q} = \sum_{i=1}^n q_i \, dx^i \otimes dx^i$, where $q_1, \ldots, q_n > 0$, yields a family of inner products on $\mathfrak{g}$. For each such Q, there is an orthonormal basis $\overline{\mathcal{B}_{\mathsf{Q}}} = \overline{\mathcal{B}}$ obtained by rescaling each basis vector in $\mathcal{B}$ by its length; elements of $\overline{\mathcal{B}}$ are

the vectors $\overline{\boldsymbol{x}_i} = \frac{1}{\|\boldsymbol{x}_i\|}\,\boldsymbol{x}_i$, for $i = 1, \ldots, n$. The structure constants for $\mathfrak{g}$ relative to the orthonormal basis $\overline{\mathcal{B}}$ are

$$\mathsf{Q}([\overline{\boldsymbol{x}}_j, \overline{\boldsymbol{x}}_k], \overline{\boldsymbol{x}}_l) = \frac{\|\boldsymbol{x}_l\|}{\|\boldsymbol{x}_j\| \cdot \|\boldsymbol{x}_k\|}\,\alpha_{jk}^l = \sqrt{\frac{q_l}{q_j q_k}}\,\alpha_{jk}^l. \tag{2}$$

Note that the set $\Lambda_{\overline{\mathcal{B}}}$ defined by $\overline{\mathcal{B}}$ is the same as the set $\Lambda_{\mathcal{B}}$ defined by $\mathcal{B}$, and hence the set of root vectors is the same for $\mathcal{B}$ and $\overline{\mathcal{B}}$. Consequently $\mathcal{B}$ and $\overline{\mathcal{B}}$ have the same root matrix Y and the same Gram matrix U.

We define the *structure vector* for $(\mathfrak{g}, \mathsf{Q})$ relative to the orthogonal basis $\mathcal{B}$ to be the $m \times 1$ vector $\boldsymbol{a}$ having as entries the squares of the nonzero structure constants relative to the orthonormal basis $\overline{\mathcal{B}}$. More precisely, the entries of $\boldsymbol{a}$ are the numbers $\frac{q_l}{q_j q_k}(\alpha_{jk}^l)^2$, for (j, k, l) in $\Lambda_{\mathcal{B}}$, listed in the dictionary order on (j, k, l).

2.2 Curvatures of nilpotent metric Lie algebras

We say that a basis $\mathcal{B} = \{\boldsymbol{x}_i\}_{i=1}^n$ of a metric Lie algebra $(\mathfrak{g}, \mathsf{Q})$ is a *Ricci-diagonal* if the Ricci form for $(\mathfrak{g}, \mathsf{Q})$ is diagonal when represented with respect to $\mathcal{B}$. The $1 \times n$ vector $\mathbf{Ric}_{\mathcal{B}}$ defined by

$$\mathbf{Ric}_{\mathcal{B}} = \left(\frac{\mathsf{ric}(\boldsymbol{x}_1, \boldsymbol{x}_1)}{\|\boldsymbol{x}_1\|^2}, \frac{\mathsf{ric}(\boldsymbol{x}_2, \boldsymbol{x}_2)}{\|\boldsymbol{x}_2\|^2}, \ldots, \frac{\mathsf{ric}(\boldsymbol{x}_n, \boldsymbol{x}_n)}{\|\boldsymbol{x}_n\|^2} \right) \tag{3}$$

will be called the *Ricci vector* for $(\mathfrak{n}, \mathsf{Q})$ relative to the basis $\mathcal{B}$. Together, an orthogonal Ricci-diagonal basis $\mathcal{B}$ for $(\mathfrak{n}, \mathsf{Q})$, its Ricci vector, and the set of lengths of the basis vectors determine the inner product and Ricci form for $(\mathfrak{n}, \mathsf{Q})$.

The next theorem gives easy-to-compute formulas for the Ricci form and Ricci vector for a metric nilpotent Lie algebra $(\mathfrak{n}, \mathsf{Q})$. Recall that the Lie bracket of a metric Lie algebra $(\mathfrak{g}, \mathsf{Q})$ is encoded in the linear map $J : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ defined by $J_{\boldsymbol{x}}(\boldsymbol{y}) = \mathrm{ad}_{\boldsymbol{y}}^*\,\boldsymbol{x}$, for $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\mathfrak{g}$. The inner product $\mathsf{Q}(\cdot, \cdot) = \langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ induces an inner product on the tensor algebra of $\mathfrak{g}$, which we also denote by $\langle \cdot, \cdot \rangle$.

Theorem 2.1 (Theorems 6 and 8, [Pay])**.** *Let $(\mathfrak{n}, \mathsf{Q})$ be a nonabelian metric nilpotent Lie algebra. Then the Ricci form for $(\mathfrak{n}, \mathsf{Q})$ is given by*

$$\mathsf{ric}(\boldsymbol{x}, \boldsymbol{y}) = -\tfrac{1}{2}\langle \mathrm{ad}_{\boldsymbol{x}}, \mathrm{ad}_{\boldsymbol{y}} \rangle + \tfrac{1}{4}\langle J_{\boldsymbol{x}}, J_{\boldsymbol{y}} \rangle, \tag{4}$$

where $\boldsymbol{x}$ and $\boldsymbol{y}$ are in $\mathfrak{n}$. Let $\mathcal{B}$ be an orthogonal Ricci-diagonal basis with associated root matrix Y and structure vector $\boldsymbol{a}$. The Ricci vector for $(\mathfrak{n}, \mathsf{Q})$ relative to $\mathcal{B}$ may be written as

$$\begin{aligned} \mathbf{Ric}_{\mathcal{B}} &= -\tfrac{1}{2} \textstyle\sum_{(j,k,l) \in \Lambda_{\mathcal{B}}} \frac{q_l}{q_j q_k} (\alpha_{jk}^l)^2 (\boldsymbol{y}_{jk}^l) \\ &= -\tfrac{1}{2}\boldsymbol{a}^T Y. \end{aligned} \tag{5}$$

Remark 2.2. From Equation (4) it follows that an orthogonal basis $\mathcal{B} = \{X_i\}$ is Ricci-diagonal if the sets $\{J_{X_i}\}$ and $\{\mathrm{ad}_{X_i}\}$ are orthogonal.

Soliton inner products on nilpotent Lie algebras may be characterized algebraically by the property that the Ricci endomorphism differs from a scalar multiple of the identity automorphism by a derivation $D = \mathrm{Ric}_{\mathsf{Q}} - \beta \,\mathrm{Id}$ of the Lie algebra ([Lau01]). The constant β is called the *soliton constant* for $(\mathfrak{n}, \mathsf{Q})$. The soliton constant is always negative when $\mathfrak{n}$ is nonabelian, and the eigenvalues for D are positive and rational (See [Heb98], [Lau01]).

The next theorem gives an easily checked linear condition that is equivalent to an inner product Q on a nilpotent Lie algebra $\mathfrak{n}$ being a soliton inner product.

Theorem 2.3 (Theorem 1, [Pay])**.** *Let $(\mathfrak{n}, \mathsf{Q})$ be a nonabelian nilpotent metric Lie algebra with orthogonal Ricci-diagonal basis $\mathcal{B}$. Let U and $\boldsymbol{a}$ be the Gram matrix and the structure vector for $(\mathfrak{n}, \mathsf{Q})$ with respect to $\mathcal{B}$. Let $m = |\Lambda_{\mathcal{B}}|$. Then Q is a soliton inner product on $\mathfrak{n}$ with nilsoliton constant β if and only if $U\boldsymbol{a} = -2\beta[1]_{m\times 1}$.*

We conclude the section with some examples that illustrate the definitions and theorems presented thus far. First we will consider the indecomposable nilpotent Lie algebras in dimensions three and four.

Example 2.4. Let $\mathfrak{h}_3$ be the three-dimensional Heisenberg algebra, and let Q be an inner product on $\mathfrak{h}_3$. There exists an orthogonal basis $\mathcal{B} = \{\boldsymbol{x}_i\}_{i=1}^3$ such that $[\boldsymbol{x}_1, \boldsymbol{x}_2] = \boldsymbol{x}_3$. Then Q is of the form $\sum_{i=1}^3 q_i \, dx^i \otimes dx^i$. There is a single root vector $\boldsymbol{y}_1 = \boldsymbol{y}_{12}^3 = (1, 1, -1)$, and the Gram matrix with respect to $\mathcal{B}$ is $U = [3]$.

Set $(q_1, q_2, q_3) = (1, 1, 1)$. Then the structure vector is $\boldsymbol{a} = [1]$, and $U[\boldsymbol{a}] = 3[1]$, so by Theorem 2.3, $(\mathfrak{h}_3, \mathsf{Q})$ is soliton with soliton constant $\beta = -3/2$. By Theorem 2.1, the Ricci vector for $(\mathfrak{h}_3, \mathsf{Q})$ relative to $\mathcal{B}$ is $-\frac{1}{2}(1, 1, -1)$. Therefore, with respect to $\mathcal{B}$, the Ricci form is represented by a diagonal matrix with diagonal entries $-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}$. The derivation $D = \mathrm{Ric} + \frac{3}{2}\,\mathrm{Id}$ corresponding to the soliton inner product has eigenvectors $\boldsymbol{x}_1, \boldsymbol{x}_2$, and $\boldsymbol{x}_3$ with eigenvalues $1, 1$, and 2 respectively. The reader may check that any choice of positive q_1, q_2 and q_3 yields a soliton inner product; all such metrics are homothetic to $\mathsf{Q} = \sum_{i=1}^3 dx^i \otimes dx^i$.

Next we consider the family of four-dimensional filiform metric nilpotent Lie algebras $(\mathfrak{l}_4, \mathsf{Q})$ studied in [IJL06] (under the name A6).

Example 2.5. The nilpotent Lie algebra $\mathfrak{l}_4$ can be represented with respect to the basis $\mathcal{B} = \{\boldsymbol{x}_i\}_{i=1}^4$ so that all the Lie algebra relations are determined by the relations $[\boldsymbol{x}_1, \boldsymbol{x}_2] = \boldsymbol{x}_3$ and $[\boldsymbol{x}_1, \boldsymbol{x}_3] = \boldsymbol{x}_4$. Define a family of inner products $\mathsf{Q}(\cdot, \cdot) = \sum_{i=1}^4 q_i \, dx^i \otimes dx^i$ on $\mathfrak{l}_4$. For all such Q, there are two root vectors $\boldsymbol{y}_1 = \boldsymbol{y}_{12}^3 = (1, 1, -1, 0)$ and $\boldsymbol{y}_2 = \boldsymbol{y}_{13}^4 = (1, 0, 1, -1)$, and the Gram matrix is $U = \left(\begin{smallmatrix} 3 & 0 \\ 0 & 3 \end{smallmatrix}\right)$.

If we define the inner product $\mathsf{Q}^\star$ by setting $(q_1, q_2, q_3, q_4) = (1, 1, 1, 1)$, then the structure vector for $(\mathfrak{l}_4, \mathsf{Q}^\star)$ relative to $\mathcal{B}$ is $\boldsymbol{a} = [1]_{2\times 1}$, and $U\boldsymbol{a} = 3[1]_{2\times 1}$, so by Theorem 2.3, $(\mathfrak{n}, \mathsf{Q})$ is soliton with soliton constant $\beta = -3/2$. The Ricci vector is

$$\mathbf{Ric} = -\tfrac{1}{2}(\boldsymbol{y}_{12}^3 + \boldsymbol{y}_{13}^4) = \tfrac{1}{2}(-2, -1, 0, 1)$$

and the corresponding derivation $D = \mathrm{Ric} + \frac{3}{2}\,\mathrm{Id}$ has eigenvalues $\frac{1}{2}, \frac{2}{2}, \frac{3}{2}$, and $\frac{4}{2}$.

In the next example we consider the five-dimensional Heisenberg Lie algebra.

Example 2.6. Let $(\mathfrak{h}_5, \mathsf{Q})$ be the five-dimensional Heisenberg Lie algebra endowed with an inner product Q. It is possible to choose an orthogonal basis $\mathcal{B} = \{\boldsymbol{x}_i\}_{i=1}^5$ such that all the Lie algebra relations are determined by

$$[\boldsymbol{x}_1, \boldsymbol{x}_2] = \boldsymbol{x}_5 \qquad \text{and} \qquad [\boldsymbol{x}_3, \boldsymbol{x}_4] = \boldsymbol{x}_5.$$

The two root vectors are $\boldsymbol{y}_{12}^5 = (1, 1, 0, 0, -1)$ and $\boldsymbol{y}_{34}^5 = (0, 0, 1, 1, -1)$, and the Gram matrix U is $\left(\begin{smallmatrix} 3 & 1 \\ 1 & 3 \end{smallmatrix}\right)$. If $\mathsf{Q} = \sum_{i=1}^5 q_i\, dx^i \otimes dx^i$, the Ricci vector is

$$\mathbf{Ric} = -\tfrac{1}{2}\left(\tfrac{q_5}{q_1 q_2}(1, 1, 0, 0, -1) + \tfrac{q_5}{q_3 q_4}(0, 0, 1, 1, -1)\right).$$

Setting $q_1 = q_2 = \cdots = q_5 = 1$ yields a soliton inner product with structure vector $(1, 1)^T$ and Ricci vector $\frac{1}{2}(-1, -1, -1, -1, 2)$.

2.3 The Ricci flow on the space of metric Lie algebras

Isometries are preserved under the Ricci flow, so that as an initial metric g_0 on a manifold M evolves under the Ricci flow, the metric g_t at time t has the isometry group $\mathrm{Iso}(g_0)$ of the initial metric as a subgroup of its isometry group $\mathrm{Iso}(g_t)$. Therefore, the set of homogeneous metrics on M is invariant under the Ricci flow.

Let (G, g) be an n-dimensional simply connected Lie group endowed with left-invariant metric g and with corresponding metric Lie algebra $(\mathfrak{g}, \mathsf{Q})$. Since the group G acts simply transitively on itself by isometries, there exists an G-invariant global framing of G. The Ricci flow at any point can therefore be expressed in terms of the Ricci flow at the tangent space to the identity. We view the Ricci flow as a flow ϕ_t on the space $\mathcal{P}^+(\mathfrak{g})$ of inner products on the vector space $\mathfrak{g}$. We will call this flow the *Ricci flow* for the Lie algebra $\mathfrak{g}$, and we will write the solution for the initial condition Q_0 as Q_t.

The Ricci flow can be thought of as a flow on the space of metric Lie algebras (modulo isometry). We now describe the structure of that space. After fixing a basis for a Lie algebra $\mathfrak{g}$ of dimension n, the Lie algebra can be identified with a

point μ in $\Lambda^2 V^* \otimes V$, where V is an n-dimensional vector space. We use $\mathfrak{g}_\mu$ to denote the algebra with underlying space V and its multiplication defined by μ.

A metric Lie algebra $(\mathfrak{g}, \mathsf{Q})$ of dimension n is a point in

$$\mathcal{X}_n = \{\mu \in \Lambda^2 V^* \otimes V : \mathfrak{g}_\mu \text{ is a Lie algebra}\} \times \mathcal{P}^+(V),$$

while a metric nilpotent Lie algebra of dimension n is a point in

$$\mathcal{N}_n = \{\mu \in \Lambda^2 V^* \otimes V : \mathfrak{g}_\mu \text{ is a nilpotent Lie algebra}\} \times \mathcal{P}^+(V).$$

Note that the Jacobi identity and the nilpotency conditions for a Lie algebra $\mathfrak{g}_\mu$ are both polynomial constraints on the structure constants so these sets are algebraic subsets.

Define an equivalence relation $\sim$ on $\mathcal{X}_n$ and $\mathcal{N}_n$ such that $(\mathfrak{g}_\mu, \mathsf{Q}_1) \sim (\mathfrak{g}_\nu, \mathsf{Q}_2)$ if and only if $(\mathfrak{g}_\mu, \mathsf{Q}_1)$ and $(\mathfrak{g}_\nu, \mathsf{Q}_2)$ are isometric. Then $\widetilde{\mathcal{X}_n} = \mathcal{X}_n/\!\sim$ parametrizes the spaces of metric Lie algebras, and $\widetilde{\mathcal{N}_n} = \mathcal{N}_n/\!\sim$ parametrizes the spaces of metric nilpotent Lie algebras.

E. Wilson showed that the simply connected nilmanifolds corresponding to metric Lie algebras $(\mathfrak{n}_\mu, \mathsf{Q}_1)$ and $(\mathfrak{n}_\nu, \mathsf{Q}_2)$ are isometric if and only if there is an isometric isomorphism mapping one to the other ([Wil82]). Hence, the space $\widetilde{\mathcal{N}_n}$ can be identified with the quotient space for the natural action of $O(n)$ on $\mathcal{N}_n$. This action is the restriction of the natural action of $GL_n(\mathbb{R})$ on $\Lambda^2 V^* \otimes V \times \mathcal{P}^+(V)$, and is defined as follows: for g in $GL_n(\mathbb{R})$,

$$g(\mathfrak{g}_\mu, \mathsf{Q}) = (\mathfrak{g}_{g\mu}, g\mathsf{Q}),$$

where $g\mu \in \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g}$ is given by

$$(g\mu)(\boldsymbol{x}, \boldsymbol{y}) = g\mu(g^{-1}\boldsymbol{x}, g^{-1}\boldsymbol{y}),$$

and the inner product $g\mathsf{Q}$ satisfies $g\mathsf{Q}(\boldsymbol{x}, \boldsymbol{y}) = \mathsf{Q}(g\boldsymbol{x}, g\boldsymbol{y})$.

2.4 The projectivized Ricci flow on the space of volume-normalized metric Lie algebras

Let $\mathcal{P}^+(\mathfrak{g})/\!\sim$ denote the space of volume-normalized inner products on a Lie algebra $\mathfrak{g}$, obtained through the equivalence relation $\mathsf{Q} \sim \lambda\mathsf{Q}$ for Q in $\mathcal{P}^+(\mathfrak{g})$ and λ in $\mathbb{R}^+$. Denote the equivalence class of inner product Q by $\overline{\mathsf{Q}}$. The Ricci flow ϕ_t for the Lie algebra $\mathfrak{g}$ projects to a flow

$$\overline{\phi}_t : \mathcal{P}^+(\mathfrak{g})/\!\sim \to \mathcal{P}^+(\mathfrak{g})/\!\sim$$

on $\mathcal{P}^+(\mathfrak{g})/\sim$ because $\phi_t(\lambda \mathsf{Q}) = \lambda \phi_t(\mathsf{Q})$ for any Q in $\mathcal{P}^+(\mathfrak{g})$ and any $\lambda > 0$. We call this flow the *projectivized Ricci flow* for $\mathfrak{g}$, and we will write $\overline{\phi}_t(\overline{\mathsf{Q}}_0)$ as $\overline{\mathsf{Q}}_t$, for $\overline{\mathsf{Q}}_0$ in $\mathcal{P}^+(\mathfrak{g})/\sim$. The space $\mathcal{P}^+(\mathfrak{g})/\sim$ has a natural closure, the compact set $\mathcal{P}^{\geq 0}(\mathfrak{g})/\sim$, where $\mathcal{P}^{\geq 0}(\mathfrak{g})$ is the set of nontrivial positive semidefinite symmetric bilinear forms on $\mathfrak{g}$, and the equivalence relation $\sim$ is defined as before.

We say that a metric Lie algebra $(\mathfrak{g}, \mathsf{Q})$ *collapses* under the Ricci flow if the limit $\overline{\mathsf{Q}}_\infty = \lim_{t \to \infty} \overline{\mathsf{Q}}_t$ exists in $\mathcal{P}^{\geq 0}(\mathfrak{g})/\sim$ but is in the boundary of $\mathcal{P}^+(\mathfrak{g})/\sim$; that is, any representative Q_∞ for the limiting normalized symmetric bilinear form $\overline{\mathsf{Q}}_\infty$ is not positive definite. We will then say that Q and $\overline{\mathsf{Q}}$ *collapse to* $\overline{\mathsf{Q}}_\infty$.

Let $\mathcal{X}_n$ and $\mathcal{N}_n$ be as defined previously. Define a new equivalence relation $\sim$ on $\mathcal{X}_n \setminus (\{0\} \times \mathcal{P}^+(V))$ and $\mathcal{N}_n \setminus (\{0\} \times \mathcal{P}^+(V))$ so that $(\mathfrak{g}_\mu, \mathsf{Q}_1) \sim (\mathfrak{g}_\nu, \mathsf{Q}_2)$ if and only if $(\mathfrak{g}_\mu, \mathsf{Q}_1)$ and $(\mathfrak{g}_\nu, \mathsf{Q}_2)$ are homothetic. Let

$$\overline{\mathcal{X}_n} = (\mathcal{X}_n \setminus (\{0\} \times \mathcal{P}^+(V)))/\sim,$$
$$\overline{\mathcal{N}_n} = (\mathcal{N}_n \setminus (\{0\} \times \mathcal{P}^+(V)))/\sim .$$

These spaces parametrize the families of nonabelian volume-normalized metric Lie algebras and nonabelian volume-normalized nilpotent metric Lie algebras, respectively. We will represent the equivalence class of $(\mathfrak{g}, \mathsf{Q})$ in $\overline{\mathcal{X}_n}$ or $\overline{\mathcal{N}_n}$ by $\overline{(\mathfrak{g}, \mathsf{Q})}$. Note that $(\mathfrak{g}_\mu, \mathsf{Q}) \sim (\mathfrak{g}_{\lambda\mu}, \mathsf{Q})$, for $\mu \in \Lambda^2 \mathfrak{g}^\star \otimes \mathfrak{g}$, $\lambda > 0$, and Q in $\mathcal{P}^+(\mathbb{R}^n)$.

We extend the projectivized Ricci flow for a single Lie algebra to a flow, also denoted by $\overline{\phi_t}$, on the spaces $\overline{\mathcal{X}_n}$ and $\overline{\mathcal{N}_n}$. It is defined by

$$\overline{\phi_t}\left(\overline{(\mathfrak{g}, \mathsf{Q})}\right) = \overline{(\mathfrak{g}, \mathsf{Q}_t)},$$

where Q_t is the solution $\phi_t(\mathsf{Q})$ to the Ricci flow for the nilpotent Lie algebra $\mathfrak{g}$ with initial condition Q. The flow is well-defined since the Ricci flow commutes with homotheties. For a metric Lie algebra $(\mathfrak{g}, \mathsf{Q})$, $\overline{\phi_t}\left(\overline{(\mathfrak{g}, \mathsf{Q})}\right)$ can be identified with $(\mathfrak{g}, \overline{\phi_t}(\overline{\mathsf{Q}}))$, where $\overline{\phi_t}$ is the projectivized Ricci flow for the single Lie algebra $\mathfrak{g}$. Note the the algebraic structure is constant under the flow: for all finite $t \geq 0$, the underlying nilpotent Lie algebras for $\overline{(\mathfrak{g}, \mathsf{Q})}$ and for $\psi_t(\overline{(\mathfrak{g}, \mathsf{Q})})$ are isomorphic.

The next simple example illustrates how a metric Lie algebra can collapse under the Ricci flow while having a nondegenerate limit point for the projectivized Ricci flow.

Example 2.7. It was shown in [IJ92] that the Ricci flow for the three-dimensional Heisenberg Lie algebra $\mathfrak{h}_3$ is given by

$$[\mathsf{Q}_t]_{\mathcal{B}} = \begin{bmatrix} (3ct+1)^{1/3} & 0 & 0 \\ 0 & (3ct+1)^{1/3} & 0 \\ 0 & 0 & (3ct+1)^{-1/3} \end{bmatrix},$$

relative to a basis $\mathcal{B} = \{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3\}$. The positive constant c is the structure constant α_{12}^3 for $\mathcal{B}$.

All inner products Q on $\mathfrak{h}_3$ are homothetic, so the space $\overline{\mathcal{N}_3}$ consists of a single point: the equivalence class of $(\mathfrak{h}_3, \mathsf{Q}_0)$. At all finite times, the metric nilpotent Lie algebras $(\mathfrak{h}_3, \mathsf{Q}_t)$ that are solutions to the Ricci flow with initial condition Q_0 project to the same point $\overline{(\mathfrak{h}_3, \mathsf{Q}_0)}$ in $\overline{\mathcal{N}_3}$. Therefore, the point $\overline{(\mathfrak{h}_3, \mathsf{Q}_0)}$ is a fixed point for the projectivized Ricci flow on $\overline{\mathcal{N}_3}$.

Yet the inner product Q_0 collapses under the flow, because

$$\lim_{t\to\infty} \frac{1}{(3ct+1)^{1/3}} \begin{bmatrix} (3ct+1)^{1/3} & 0 & 0 \\ 0 & (3ct+1)^{1/3} & 0 \\ 0 & 0 & (3ct+1)^{-1/3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

forcing $\lim_{t\to\infty} \overline{\mathsf{Q}_t}$ to be in the boundary of $\mathcal{P}^+(\mathfrak{h}_3)/\!\sim$.

2.5 The Lie bracket flow and the projectivized Lie bracket flow

Let $(\mathfrak{g}, \mathsf{Q})$ be a metric Lie algebra, and let $\mathcal{B} = \{\boldsymbol{x}_i\}_{i=1}^n$ be an orthonormal basis for $\mathfrak{g}$, with respect to which Q is written as $\sum_{i=1}^n dx^i \otimes dx^i$. We will say that $\mathcal{B}$ is a *stably Ricci-diagonal basis* if $\mathcal{B}$ is Ricci-diagonal for all metrics $\mathsf{Q}' = \sum_{i=1}^n c_i dx^i \otimes dx^i$, with $c_i > 0$ for $i = 1, \ldots, n$. It is often possible to find stably Ricci-diagonal bases. All three-dimensional unimodular metric Lie algebras and many four-dimensional metric Lie algebras have such bases ([IJ92], [IJL06]).

If the basis $\mathcal{B}$ is a stably Ricci-diagonal basis for $(\mathfrak{g}, \mathsf{Q})$, both the inner product and the Ricci form remain diagonal with respect to $\mathcal{B}$ under the Ricci flow, and the positive functions $q_1, q_2, \ldots, q_n$ encode the solution $\mathsf{Q}_t = \sum_{i=1}^n q_i\, dx^i \otimes dx^i$ to the Ricci flow for $\mathfrak{g}$ with initial condition $\mathsf{Q}_0 = \mathsf{Q}$. For $1 \leq j, k, l \leq n$, define the function a_{jk}^l of t by

$$a_{jk}^l = \frac{q_l}{q_j q_k} (\alpha_{jk}^l)^2, \tag{6}$$

where α_{jk}^l is the structure constant for $(\mathfrak{g}, \mathsf{Q}_0)$ relative to the Q_0-orthonormal basis $\mathcal{B}$ as defined in Equation (1). These functions represent structure constants for $(\mathfrak{g}, \mathsf{Q}_t)$ relative to the Q_t-orthonormal basis $\overline{\mathcal{B}_{\mathsf{Q}_t}}$ defined in Section 2.1.

Let $a_1, \ldots, a_m$ be an enumeration of such functions a_{jk}^l for (j, k, l) in $\Lambda_{\mathcal{B}}$, in dictionary order. Define the *structure vector function* $\boldsymbol{a} : \mathbb{R} \to \mathbb{R}^m$ by

$$\boldsymbol{a}_t = \boldsymbol{a}(t) = (a_1(t), \ldots, a_m(t))^T \tag{7}$$

for each t; this is just the $m \times 1$ structure vector relative to the basis $\overline{\mathcal{B}_{\mathsf{Q}_t}}$ for $(\mathfrak{g}, \mathsf{Q}_t)$. Observe that the vector $\boldsymbol{a}_t$ is positive at all times $t \geq 0$. We call the flow $\boldsymbol{a}_t$ the *Lie bracket flow*.

The structure vector function $\boldsymbol{a}_t$ defined in Equation (7) takes values in $\mathbb{R}^m$. We are interested in the asymptotic behavior of $\boldsymbol{a}_t$ modulo rescaling; that is, the values of $[\boldsymbol{a}_t]$ in projective space $P^{m-1}(\mathbb{R})$. Since entries of $\boldsymbol{a}_t$ are positive for all $t > 0$, we will describe the flow of $[\boldsymbol{a}_t]$ using the homogeneous variables $s_1, \dots, s_{m-1}$ given by

$$\boldsymbol{s} = (s_1, \dots, s_{m-1}) = \left(\frac{a_1}{a_m}, \frac{a_1}{a_m}, \dots, \frac{a_{m-1}}{a_m} \right). \tag{8}$$

These coordinates parametrize the affine algebraic subset $\mathbb{A}_{m-1} = \{(s_1 : s_2 : \cdots : s_{m-1} : 1)\}$ of $P^{m-1}(\mathbb{R})$.

2.6 Limits for the Ricci flow and the Lie bracket flow as time goes to infinity

In general, if $[\boldsymbol{a}_t]$ converges to $[\boldsymbol{a}_\infty]$ under the projectivized Lie bracket flow (relative to some stably Ricci-diagonal basis), the Ricci flow does not necessarily limit on a Lie group endowed with a left-invariant metric whose corresponding metric Lie algebra has structure constants equal to $[\boldsymbol{a}_\infty]$. In general, caution must be exercised in extracting limits for the projectivized Ricci flow from limits for the projectivized Lie bracket flow. However, in the case of the projectivized Ricci flow on the space of volume-normalized metric nilpotent Lie algebras, problems do not arise: one can say that the structure constants for the metric Lie algebra associated to the limiting homogeneous space are given by $[\boldsymbol{a}_\infty]$.

It was demonstrated in [Wil82] that the isometry group of a simply connected nilmanifold N associated to metric nilpotent Lie algebra $(\mathfrak{n}, \mathsf{Q})$ is equal to the semidirect product $K \ltimes N$ of translations from N and a compact isotropy group K, and the Lie algebra of K is equal to $\mathrm{Aut}\left((\mathfrak{n}, \mathsf{Q})\right)$. Suppose that $\overline{(\mathfrak{n}, \mathsf{Q}_t)}$ converges to $\overline{(\mathfrak{n}_\infty, \mathsf{Q}_\infty)}$. If $f \in \mathrm{Aut}\left((\mathfrak{n}, \mathsf{Q}_t)\right)$ for all t, then $f \in \mathrm{Aut}\left((\mathfrak{n}_\infty, \mathsf{Q}_\infty)\right)$, where we identify f with a linear map of the vector space V used to define $\mathcal{N}_n$. In contrast, whenever $\mathfrak{n} \not\cong \mathfrak{n}_\infty$, the n-dimensional group of translational isometries for finite time does not coincide with the group of translational isometries for the limiting nilmanifold.

There is a one-to-one correspondence between Lie brackets in $\Lambda^2 V^* \otimes V$, modulo the action of $GL_n(\mathbb{R})$, and metric nilpotent Lie algebras $(\mathfrak{n}, \mathsf{Q})$ in $\mathcal{N}_n$. The space $\mathcal{N}_n$ is closed and invariant for the flow ϕ_t, and if $\boldsymbol{a}_t$ is nilpotent for all t, and $[\boldsymbol{a}_t] \to [\boldsymbol{a}_\infty]$, then $\boldsymbol{a}_\infty$ is nilpotent. Suppose that $\lim_{t\to\infty}[\boldsymbol{a}_t] = [\boldsymbol{a}_\infty]$. Then there is a corresponding limit of metric nilpotent Lie algebras $\lim_{t\to\infty} \overline{(\mathfrak{n}, \mathsf{Q}_t)} = \overline{(\mathfrak{n}_\infty, \mathsf{Q}_\infty)}$ in $\overline{\mathcal{N}_n}$. At all finite times, the geometry of the nilmanifold class corresponding to $\overline{(\mathfrak{n}, \mathsf{Q}_t)}$ is completely determined by left-multiplication of the inner product Q_t over the Lie group $N = \exp(\mathfrak{n})$, while the geometry of the nilmanifold class corresponding to $\overline{(\mathfrak{n}_\infty, \mathsf{Q}_\infty)}$ is completely determined by left-multiplication of the inner product Q_∞ over the Lie group $N_\infty = \exp(\mathfrak{n}_\infty)$. Hence, the local geometry (modulo homothety),

of the nilmanifolds for $\overline{(\mathfrak{n}, \mathsf{Q}_t)}$ converges to the local geometry (modulo homothety) for the nilmanifold class defined by $\overline{(\mathfrak{n}_\infty, \mathsf{Q}_\infty)}$.

Observe that the vector $\boldsymbol{a}_t$ determines the metric nilpotent Lie algebra $(\mathfrak{n}, \mathsf{Q}_t)$ up to isometry because it tells us the Lie algebra structure's structure constants relative to an orthonormal basis. Hence the Lie bracket flow determines the Ricci flow and the two flows are equivalent.

Suppose that one could solve for $\boldsymbol{a}_t$ exactly. Equation (5) describes the Ricci vector in terms of $\boldsymbol{a}_t$, which in turn gives the Ricci form $\mathsf{ric}_{\mathsf{Q}_t}$ as a function of time. Using the equation for the Ricci flow, one can solve for the functions $q_1, \ldots, q_m$ giving the inner product Q_t as a function of time by integrating $-2\mathsf{ric}_{\mathsf{Q}_t}$ with respect to t. Also note that since the connection and curvatures for $(\mathfrak{n}, \mathsf{Q}_t)$ depend only on the structure constants relative to an orthonormal basis, these geometric quantities are determined by the Lie bracket flow. In order to compute or estimate geometric quantities, it is sufficient to consider only the Lie bracket flow, which in many cases is easier to work with than the Ricci flow.

3 Systems of ordinary differential equations

In this section, we set up systems of ODE's for the Ricci flow and for the projectivized bracket flow.

3.1 O.D.E.'s for the Ricci flow and the Lie bracket flow

The next theorem describes how the structure vector function $\boldsymbol{a}_t$ evolves under the Ricci flow, assuming the existence of a stably Ricci-diagonal basis.

Theorem 3.1. *Let $(\mathfrak{g}, \mathsf{Q})$ be a metric Lie algebra with stably Ricci-diagonal basis $\mathcal{B} = \{\boldsymbol{x}_i\}_{i=1}^n$. Let Y be the root matrix and let U be the Gram matrix for $(\mathfrak{n}, \mathsf{Q})$ relative to $\mathcal{B}$. Let Q_t be the solution to the Ricci flow for $\mathfrak{g}$ with initial condition $\mathsf{Q}_0 = \mathsf{Q}$, and let the functions $q_1, \ldots, q_n$ be defined by $\mathsf{Q}_t = \sum_{i=1}^n q_i \, dx^i \otimes dx^i$. Let $\boldsymbol{a}_t = (a_i(t))$ be the structure vector function for $(\mathfrak{g}, \mathsf{Q}_t)$ as defined in Equation (7). Then*

$$\left(\frac{q_1'}{q_1}, \ldots, \frac{q_n'}{q_n} \right) = -2\, \mathbf{Ric}_{\mathcal{B}}, \tag{9}$$

and

$$\left(\frac{a_1'}{a_1}, \frac{a_2'}{a_2}, \ldots, \frac{a_m'}{a_m} \right) = 2\, \mathbf{Ric}_{\mathcal{B}} Y^T. \tag{10}$$

In the case that $\mathfrak{g}$ is nonabelian and nilpotent,

$$\left(\frac{q_1'}{q_1}, \ldots, \frac{q_n'}{q_n} \right) = \boldsymbol{a}^T Y, \tag{11}$$

and

$$\left(\frac{a_1'}{a_1}, \frac{a_2'}{a_2}, \ldots, \frac{a_m'}{a_m}\right) = -\boldsymbol{a}^T U. \tag{12}$$

Remark 3.2. Observe that the O.D.E.'s for $a_1, a_2, \ldots, a_m$ are quadratic, and they can always be put in log-linear form: for example, Equation (10) becomes $(\ln a_i)' = -\sum_{j=1}^m u_{ij} a_j$, for $i = 1, \ldots, m$.

Remark 3.3. The functions a_i' are smooth and locally bounded on $\mathbb{R}^m$; hence solutions for $a_i(t)$ exist for all $t > 0$. By the same reasoning, solutions to $q_i(t)$ exist for all time. However they are not guaranteed to be positive for all time: for example, the round metric on $SU(2)$ becomes zero in finite time ([IJ92]).

Proof. With respect to the basis $\mathcal{B}$, the inner product is represented by the diagonal matrix

$$[\mathsf{Q}]_{\mathcal{B}} = \operatorname{diag}(q_1, q_2, \ldots, q_n),$$

and the Ricci form is represented by a diagonal matrix $[\mathsf{ric}]_{\mathcal{B}}$. If we rewrite the Ricci form with respect to the orthonormal basis $\overline{\mathcal{B}}$ obtained by rescaling $\mathcal{B}$, we get

$$[\mathsf{ric}]_{\mathcal{B}} = [\mathsf{Q}]_{\mathcal{B}} [\mathsf{ric}]_{\overline{\mathcal{B}}}.$$

By definition of the Ricci vector, the matrix $[\mathsf{ric}]_{\overline{\mathcal{B}}}$ is the diagonal matrix whose diagonal entries are the entries of the Ricci vector $\mathbf{Ric}_{\mathcal{B}}$ as defined in Equation (3). By equating the diagonal entries of the matrices on both sides of the matrix equation $[\mathsf{Q}_t]_{\mathcal{B}}' = -2[\mathsf{ric}_{\mathsf{Q}_t}]_{\mathcal{B}}$, the Ricci flow may then be written

$$\left(\frac{q_1'}{q_1}, \ldots, \frac{q_n'}{q_n}\right) = -2\,\mathbf{Ric}_{\mathcal{B}}. \tag{13}$$

Changing variables to $\ln q_i$, for $i = 1, \ldots, m$, yields

$$(\ln q_1, \ldots, \ln q_n)' = -2\,\mathbf{Ric}_{\mathcal{B}}. \tag{14}$$

Now we compute the derivative of the functions $\ln a_i$, for $i = 1, \ldots, m$, defined in Equation (6), temporarily switching to the indexing $a_{jk}^l, (i,j,k) \in \Lambda_{\mathcal{B}}$, for the functions $a_i, i = 1, \ldots, m$. For $(i,j,k) \in \Lambda_{\mathcal{B}}$, the derivative of $\ln(a_{jk}^l)$ is

$$\begin{aligned}\left(\ln(a_{jk}^l)\right)' &= \left(\ln\left((\alpha_{jk}^l)^2 \frac{q_l}{q_j q_k}\right)\right)' \\ &= -(\ln q_j)' - (\ln q_k)' + (\ln q_l)' \\ &= -\,(\ln q_1, \ldots, \ln q_n)' \cdot \boldsymbol{y}_{jk}^l.\end{aligned}$$

Using Equation (14) to rewrite the right side of the previous line, we get

$$\left(\ln(a^l_{jk})\right)' = 2\,\mathbf{Ric}_{\mathcal{B}}(\boldsymbol{y}^l_{jk})^T$$

These m linear equations conjoin to become Equation (10). If $\mathfrak{g}$ is nilpotent, by Equation (5) of Theorem 2.1, $\mathbf{Ric}_{\mathcal{B}} = -\frac{1}{2}\boldsymbol{a}^T Y$. Substitution of this into Equations (9) and (10) and the identity $YY^T = U$ yield Equations (11) and (12). □

To illustrate Theorem 3.1, we revisit some of the previous examples.

Example 3.4. For $(\mathfrak{h}_3, \mathsf{Q})$ as in Example 2.4, if we let $a_1 = \frac{q_3}{q_1 q_2}$, Equation (12) reduces to the single equation

$$a_1' = -3a_1^2.$$

Integrating, we get $a_1(t) = (3t + c)^{-1}$, where $1/c$ is the structure constant α^3_{12} for the initial orthonormal basis. Then Equation (11) is

$$(\ln q_1, \ln q_2, \ln q_3)' = (3t + c)^{-1}(1, 1, -1),$$

and integration gives the solutions q_1, q_2, q_3 already presented in Example 2.7.

For $(\mathfrak{l}_4, \mathsf{Q})$ as in Example 2.5, when we let $a_1 = \frac{q_3}{q_1 q_2}$ and $a_2 = \frac{q_4}{q_1 q_3}$, the system of equations from Equation (12) is

$$a_1' = -3a_1^2, \qquad a_2' = -3a_2^2,$$

which decouples, so it is not hard to solve for

a_1 and a_2, and then q_1, q_2, q_3 and q_4. For the metric Heisenberg algebra $(\mathfrak{h}_5, \mathsf{Q})$ as in Example 2.6, after letting $a_1 = \frac{q_5}{q_1 q_2}$ and $a_2 = \frac{q_5}{q_3 q_4}$, we get the system

$$\begin{aligned} a_1' &= -3a_1^2 - a_1 a_2 \\ a_2' &= -a_1 a_2 - 3a_2^2, \end{aligned} \tag{15}$$

which, in contrast to the first two examples, has no simple explicit general solution. Instead, it is necessary to make a qualitative analysis of a projectivization of such a system; this is the goal of the next section.

3.2 O.D.E.'s for the projectivized Lie bracket flow

Now we define some matrices and hyperplanes that are needed to state the main theorem of the section. Define the $(m-1) \times m$ matrix P by

$$P = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 1 & 0 & \cdots & 0 & -1 \\ 0 & 0 & 1 & & 0 & -1 \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \end{bmatrix}.$$

Let U be an $m \times m$ matrix. For $i = 1, \ldots, m-1$, define the vector $\boldsymbol{n}_i$ by $\boldsymbol{n}_i = \boldsymbol{e}_i^T PU$, where $\{\boldsymbol{e}_i\}_{i=1}^{m-1}$ is the standard orthonormal basis for $\mathbb{R}^m$. The vector $\boldsymbol{n}_i$ is the ith row of the matrix U minus the mth row of the matrix U.

The vectors $\boldsymbol{n}_1, \ldots, \boldsymbol{n}_{m-1}$ define a set of $m-1$ hyperplanes $\mathcal{H}_1^0, \mathcal{H}_2^0, \ldots, \mathcal{H}_{m-1}^0$, where we let

$$\mathcal{H}_i^0 = \boldsymbol{n}_i^\perp, \quad \text{for } i = 1, \ldots, m-1. \tag{16}$$

For $i = 1, \ldots, m-1$, define the open half-spaces $\mathcal{H}_i^+$ and $\mathcal{H}_i^-$ by

$$\begin{aligned}\mathcal{H}_i^+ &= \{\boldsymbol{a} \in \mathbb{R}^m \,|\, \boldsymbol{a} \cdot \boldsymbol{n}_i > 0\} \\ \mathcal{H}_i^- &= \{\boldsymbol{a} \in \mathbb{R}^m \,|\, \boldsymbol{a} \cdot \boldsymbol{n}_i < 0\}.\end{aligned}$$

The next proposition describes some properties of these hyperplanes and their normal vectors, when U is the Gram matrix for a metric nilpotent Lie algebra $(\mathfrak{n}, \mathsf{Q})$ relative to an orthogonal basis $\mathcal{B}$.

Proposition 3.5. *Let $(\mathfrak{n}, \mathsf{Q})$ be a nonabelian metric nilpotent Lie algebra with orthogonal Ricci-diagonal basis $\mathcal{B}$. If U is the Gram matrix for $(\mathfrak{n}, \mathsf{Q})$ with respect to $\mathcal{B}$, then the intersection $\cap_{i=1}^{m-1} \mathcal{H}_i^0$ of the hyperplanes $\mathcal{H}_1^0, \mathcal{H}_2^0, \ldots, \mathcal{H}_m^0$ is equal to*

$$\ker PU = \{\boldsymbol{v} \,:\, U\boldsymbol{v} = \lambda [1]_{m \times 1} \quad \textit{for some } \lambda \textit{ in } \mathbb{R}\}.$$

If in addition, $\alpha_{jk}^j \neq 0$ for all $1 \leq j, k \leq n$, then for all $i = 1, \ldots, m-1$,

1. *the ith entry of the vector $\boldsymbol{n}_i$ is positive and the mth entry of $\boldsymbol{n}_i$ is negative, and*
2. *the point $(0, 0, \ldots, 0, 1)$ lies in the open half space $\mathcal{H}_i^-$.*

Note that the subspace $\ker PU$ of $\mathbb{R}^m$ is always nontrivial because $\operatorname{rank} PU = m-1$.

Proof. By the definitions of the vectors $\boldsymbol{n}_i$ and the hyperplanes $\mathcal{H}_i^0$, a vector $\boldsymbol{v}$ lies in $\ker PU$ if and only if $\boldsymbol{v}$ is in $\mathcal{H}_i^0$ for all $i = 1, \ldots, m-1$. For a vector $\boldsymbol{v}$ in $\mathbb{R}^m$, $PU\boldsymbol{v} = \boldsymbol{0}$ if and only if the ith entry and the mth entry of the vector $U\boldsymbol{v}$ are the same real number λ, for $i = 1, \ldots, m-1$. This is true if and only if $U\boldsymbol{v} = \lambda[1]_{m \times 1}$. Therefore, the kernel of the matrix PU is spanned by vectors $\boldsymbol{v}$ so that $U\boldsymbol{v}$ is a scalar multiple of $[1]$. This proves the first part of the proposition.

To prove Statements 1 and 2, simply use the definition of PU and that when $\alpha_{jk}^j \neq 0$ for all j and k, the diagonal entries of $U = (u_{ij})$ are all three, while the off-diagonal entries are all in the set $\{-2, -1, 0, 1, 2\}$. Then

$$(0, 0, \ldots, 0, 1) \cdot \boldsymbol{n}_i = (\boldsymbol{n}_i)_m = u_{im} - 3 < 0$$

for $i = 1, \ldots, m-1$. $\square$

The next theorem describes the evolution of the structure vector $\boldsymbol{a}_t$ for $(\mathfrak{n}, \mathsf{Q}_t)$, modulo rescaling, as the metric nilpotent Lie algebra $(\mathfrak{n}, \mathsf{Q})$ evolves under the Ricci flow. The hyperplanes $\mathcal{H}_1^0, \mathcal{H}_2^0, \ldots, \mathcal{H}_{m-1}^0$ defined in Equation (16) are homogeneous sets in $\mathbb{R}^m$. For all $i = 1, \ldots, m$, $\mathcal{H}_i^0 \setminus \{\mathbf{0}\}$ projects to an algebraic set $[\mathcal{H}_i^0]$ of codimension one in $P^{m-1}(\mathbb{R})$ whose intersection with $\mathbb{A}_{m-1}$ is a hyperplane. For $i = 1, \ldots, m-1$, define the functions $\eta_i : \mathbb{R}^{m-1} \to \mathbb{R}$ by

$$\eta_i(\boldsymbol{s}) = \boldsymbol{n}_i \cdot (s_1, \ldots, s_{m-1}, 1), \tag{17}$$

for $\boldsymbol{s}$ in $\mathbb{R}^{m-1}$.

Theorem 3.6. *Let $(\mathfrak{n}, \mathsf{Q})$ be a metric nonabelian nilpotent Lie algebra with basis $\mathcal{B} = \{\boldsymbol{x}_i\}_{i=1}^n$ that is stably Ricci-diagonal. Let $m = |\Lambda_{\mathcal{B}}|$, and let $\mathsf{Q}_t = \sum_{i=1}^n q_i \, dx^i \otimes dx^i$ denote the solution to the Ricci flow at time t, written relative to $\mathcal{B}$. Let the functions $a_1, \ldots, a_m$ and $s_1, \ldots, s_{m-1}$ be as defined in Equations (6) and (8) respectively. Then*

$$s_i'(t) = -a_m s_i \eta_i(\boldsymbol{s}(t)). \tag{18}$$

for $i = 1, \ldots, m-1$.

Proof of Theorem 3.6. Using the quotient rule, we get

$$s_i' = \left(\frac{a_i}{a_m}\right)' = \frac{a_i' a_m - a_m' a_i}{a_m^2}.$$

By Theorem 3.1, for $i = 1, \ldots, m-1$, the function a_i' is equal to $-a_i$ times the product of the ith row of U with $\boldsymbol{a}$, so

$$\begin{aligned} s_i' &= \frac{a_i}{a_m} \left[-(\text{row } i \text{ of } U) \cdot \boldsymbol{a} + (\text{row } m \text{ of } U) \cdot \boldsymbol{a}\right] \\ &= -s_i \left[(\text{row } i \text{ of } PU) \cdot \boldsymbol{a}\right] \\ &= -s_i a_m \left[(\text{row } i \text{ of } PU) \cdot (s_1, \ldots, s_{m-1}, 1)\right] \end{aligned}$$

Thus, $s_i'(t) = -a_m s_i \eta_i(\boldsymbol{s}(t))$, as desired. ☐

The hyperplanes $[\mathcal{H}_i^0], i = 1, \ldots, m$ divide the affine space $\mathbb{A}_{m-1}$ into chambers. The theorem implies that the general direction of a $\boldsymbol{s}_t$ trajectory depends only upon which chamber a point $\boldsymbol{s}$ in $\mathbb{A}_{m-1}$ is on, and that equilibrium points come from soliton metrics (solutions to $Uv = [1]$).

Remark 3.7. Orbits of the system in Equation (18) agree with those of the system

$$(\ln s_i)' = -\eta_i(\boldsymbol{s}). \tag{19}$$

The direction of the trajectories for the two systems is the same because $a_m > 0$. Although the system in Equation (18) may be quite difficult to solve exactly, as the functions $\eta_i, i = 1, \ldots, m$ are linear in the variables s_i, the system in Equation (19) is more tractable.

In the next example, we illustrate the definitions of the vectors $\boldsymbol{n}_i$ and hyperplanes $\mathcal{H}_i^0$, and we give an application of the previous theorem.

Example 3.8. Let $(\mathfrak{n}, \mathsf{Q})$ be the five-dimensional metric Lie algebra that with respect to an orthogonal basis $\mathcal{B} = \{\boldsymbol{x}_i\}_{i=1}^5$ has the following bracket relations:

$$[\boldsymbol{x}_1, \boldsymbol{x}_3] = \boldsymbol{x}_4, \qquad [\boldsymbol{x}_1, \boldsymbol{x}_4] = \boldsymbol{x}_5, \qquad [\boldsymbol{x}_2, \boldsymbol{x}_3] = \boldsymbol{x}_5.$$

By Theorem 2.1, $\mathcal{B}$ is a Ricci diagonal basis. By Remark 2.2, the Ricci form is diagonal for all rescalings of vectors in $\mathcal{B}$, so the basis is stably Ricci-diagonal. Then

$$\Lambda_{\mathcal{B}} = \{(1,3,4), (1,4,5), (2,3,5)\},$$

and the Gram matrix U and the matrix PU are given by

$$U = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix} \qquad \text{and} \qquad PU = \begin{bmatrix} 2 & -1 & -2 \\ -1 & 2 & -2 \end{bmatrix}.$$

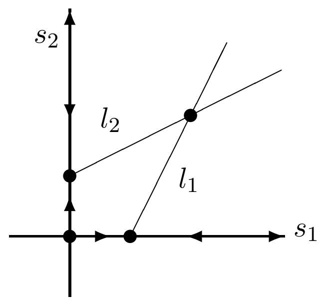
The kernel of PU is spanned by $\boldsymbol{v} = (2,2,1)^T$; note that $U\boldsymbol{v} = 7(1,1,1)^T$. Therefore, by Theorem 2.3, if the inner product Q which has $\mathcal{B}$ as an orthogonal basis has structure vector $\boldsymbol{a} = (\frac{q_4}{q_1 q_3}, \frac{q_5}{q_1 q_4}, \frac{q_5}{q_2 q_3})$ equal to $(2,2,1)^T$, then it is soliton with soliton constant $\beta = -7/2$. In that case the Ricci vector is $-\frac{1}{2}(4,1,3,0,-3)$, and the derivation $D = \mathrm{Ric} - \beta\,\mathrm{Id}$ is represented by $[D]_{\mathcal{B}}$, the diagonal matrix $\mathrm{diag}(3,6,4,7,10)$. This example will be revisited later in Example 4.3, and soliton inner products will be described.

Here, $\boldsymbol{n}_1 = (2,-1,-2)$ and $\boldsymbol{n}_2 = (-1,2,-2)$. The hyperplanes $\mathcal{H}_1^0$ and $\mathcal{H}_1^0$ are the planes $\boldsymbol{n}_1^\perp = (2a_1 - a_2 - 2a_3 = 0)$ and $\boldsymbol{n}_2^\perp = (-a_1 + 2a_2 - 2a_3 = 0)$ in $\mathbb{R}^3$. The sets $\mathcal{H}_1^0 \cap (a_3 \neq 0)$ and $\mathcal{H}_2^0 \cap (a_3 \neq 0)$ in $\mathbb{R}^3$ project to lines $l_1 = [\mathcal{H}_1^0]$ and $l_2 = [\mathcal{H}_2^0]$ in the subset $\mathbb{A}_2$ of $P^2(\mathbb{R})$. In s_1-s_2 coordinates on $\mathbb{A}_2$, the lines l_1 and l_2 are given by $2s_1 - s_2 = 2$ and $s_1 - 2s_2 = -2$ respectively, and they intersect in the point $(s_1, s_2) = (2,2)$. Therefore the lines l_1 and l_2 intersect in the point $(2:2:1)$ in $\mathbb{A}_2 \subset P^2(\mathbb{R})$, whose preimage under the map $\boldsymbol{v} \mapsto [\boldsymbol{v}]$ is the line of intersection $\mathbb{R}(2,2,1)$ of the planes $\mathcal{H}_1^0$ and $\mathcal{H}_2^0$ in $\mathbb{R}^3$. This line contains the orbit of the structure vector $\boldsymbol{a}_t$ for any soliton metric under the Ricci flow. See Figure 1.

The linear functions

$$\eta_1(s_1, s_2) = 2s_1 - s_2 - 2 \quad \text{and}$$
$$\eta_2(s_1, s_2) = -s_1 + 2s_2 - 2$$

Figure 1: The flow $\boldsymbol{s}(t)$ for Example 3.8



determine coordinates on $\mathbb{A}_2$ such that $\eta_1\left((0,0)\right) < 0$ and $\eta_2\left((0,0)\right) < 0$, and at the point of intersection $(s_1, s_2) = (2,2)$ of the lines l_1 and l_2, $(\eta_1, \eta_2) = (0,0)$.

The main features of the slope field for the flow $\boldsymbol{s}_t$ are that

- Points above l_2 move downward, points below l_2 move upward, and points on l_2 other than $(2,2)$ move horizontally; and
- Points to the left of l_1 move left, points to the right of l_1 move to the right, and points on l_1 other than $(2,2)$ move vertically.

(Compare to the arrows drawn on the axes, and Part 2 of Proposition 3.5.) It can be seen from the slope field that for any initial point $\boldsymbol{s}_0 > 0$, eventually the trajectory enters the cone $\eta_1(\boldsymbol{s})\eta_2(\boldsymbol{s}) > 0$, and once inside that cone, asymptotically approaches $(2,2)$.

We note that the fixed points on the axes correspond to soliton metric nilpotent Lie algebras whose underlying algebra is not isomorphic to $\mathfrak{n}$.

3.3 Invariants for the Ricci flow

In previous studies of the Ricci flow for homogeneous spaces, the systems of ODE's for the Ricci flow are solved by finding invariant quantities under the flow. The next proposition gives an easy way to find some of these invariant quantities: vectors in the kernel of the root matrix Y for $(\mathfrak{n}, \mathsf{Q})$ relative to orthogonal Ricci-diagonal basis $\mathcal{B}$ yield conserved monomial quantities for the Ricci flow for $\mathfrak{n}$.

Proposition 3.9. *Let $(\mathfrak{n}, \mathsf{Q})$ be a nonabelian metric nilpotent Lie algebra with orthogonal Ricci-diagonal basis $\mathcal{B}$. Suppose that there is a stably Ricci-diagonal basis with respect to which the solution Q_t to the Ricci flow is expressed as $\mathsf{Q}_t = \sum_{i=1}^n q_i\, dx^i \otimes dx^i$. Let Y be the root matrix for $(\mathfrak{n}, \mathsf{Q})$ relative to $\mathcal{B}$. Let $\boldsymbol{a}_t$ be as defined in Equation (6).*

The constant vector $\boldsymbol{d} = (d_1, \ldots, d_n)$ satisfies the condition that $\boldsymbol{d}Y^T \boldsymbol{a}_t = \boldsymbol{0}$ for all t if and only if the quantity $q_1^{d_1} q_2^{d_2} \cdots q_n^{d_n}$ is preserved under the Ricci flow. In particular, if $\boldsymbol{d}Y^T = \boldsymbol{0}$ then $q_1^{d_1} q_2^{d_2} \cdots q_n^{d_n}$ is preserved.

Proof. The quantity $q_1^{d_1} q_2^{d_2} \cdots q_n^{d_n}$ is preserved if and only if its natural logarithm $\sum_{i=1}^n d_i \ln q_i$ is preserved. This is true if and only if for all $t > 0$,

$$\begin{aligned} 0 &= \frac{d}{dt} \sum_{i=1}^n d_i (\ln q_i) \\ &= \sum_{i=1}^n d_i (\ln q_i)' \\ &= (d_1, \ldots, d_n)(\ln q_1, \ldots, \ln q_n)'^T. \end{aligned}$$

Rewriting $(\ln q_1, \ldots, \ln q_n)'$ using Equation (11), we see that $q_1^{d_1} q_2^{d_2} \cdots q_n^{d_n}$ is preserved if and only if

$$(d_1, \ldots, d_n) Y^T \boldsymbol{a} = 0.$$

Clearly, if $\boldsymbol{d}Y^T = \boldsymbol{0}$ then $q_1^{d_1} q_2^{d_2} \cdots q_n^{d_n}$ is preserved. $\square$

Example 3.10. For $(\mathfrak{h}_3, \mathsf{Q})$ as in Example 2.4, conserved quantities $q_1 q_3$, $q_2 q_3$ and q_1/q_2 for the Ricci flow for $(\mathfrak{h}_3, \mathsf{Q})$ come from the vectors $(1,0,1)^T, (0,1,1)^T$ and $(1,-1,0)^T$ in the kernel of the root matrix $Y = (1, 1, -1)$.

3.4 The phase portrait for the projectivized Lie bracket flow

The structure vector $\boldsymbol{a}_t$ for a metric nilpotent Lie algebra $(\mathfrak{n}, \mathsf{Q})$ evolves under the Ricci flow according to the law in Equation (12). The projection of the $\boldsymbol{a}_t$ flow on $\mathbb{R}^m$ to the flow $[\boldsymbol{a}_t]$ on $P^{m-1}(\mathbb{R})$, as represented by the coordinates $\boldsymbol{s}_t$ in $\mathbb{A}_{m-1}$, obeys Equation (18); and by Remark 3.7, has the same trajectories as the solutions to Equation (19). We describe the properties of a flow satisfying Equation (19).

Lemma 3.11. *Let U be the Gram matrix for a nonabelian nilpotent metric Lie algebra $(\mathfrak{n}, \mathsf{Q})$ with respect to an orthogonal Ricci-diagonal basis $\mathcal{B}$, with $|\Lambda_{\mathcal{B}}| = m$. Let the functions η_i and hyperplanes $[\mathcal{H}_i^0]$, for $i = 1, \ldots, m-1$, be as defined in Section 3. Then the system of ordinary differential equations*

$$(\ln s_i)' = -\eta_i(\boldsymbol{s}_t), \quad i = 1, \ldots, m-1,$$

has the following properties.

1. *The sets $\partial(\boldsymbol{s} > 0)$ and $(\boldsymbol{s} > 0)$ are invariant under the flow.*

2. *The set of equilibrium solutions in* $(\boldsymbol{s} \geq 0)$ *is nonempty and compact, and is equal to the union* $\mathcal{S} = \cup \mathcal{S}_M$ *of all sets of the form*

$$\mathcal{S}_M = \bigcap_{i \notin M} [\mathcal{H}_i^0] \cap \bigcap_{i \in M} (s_i = 0) \cap (\boldsymbol{s} \geq 0),$$

   where M *varies over all subsets of* $\{1, \ldots, m-1\}$.

3. *Define the subset* $\mathcal{S}^-$ *of* $\mathcal{S}$ *to be the union of the points* $\boldsymbol{b} = (b_i)$ *in* $\mathcal{S}$ *such that there exists* i *such that* $b_i = 0$ *and* $\eta_i(\boldsymbol{b}) < 0$. *All points in* $\mathcal{S}^-$ *repel nearby points in* $(\boldsymbol{s} > 0)$.

We remark that it is possible to have continuous families of fixed points in $(\boldsymbol{s} > 0)$ for the system in Equation (19) (see Examples 27 and 28 and Theorem 29 from [Pay]).

Proof. The vector function $\boldsymbol{s}_t$ evolves according to law $\boldsymbol{s}' = \boldsymbol{F}(\boldsymbol{s})$, where the vector field $\boldsymbol{F} = (F_i)$ is defined by $F_i(\boldsymbol{s}) = -s_i \eta_i(\boldsymbol{s})$ for $i = 1, \ldots, m$. Clearly $\boldsymbol{F}$ is tangent to the boundary of $(\boldsymbol{s} > 0)$, so that the boundary is an invariant set. Since solutions exist everywhere and are unique, the interior $(\boldsymbol{s} > 0)$ is forced to be invariant under the flow also. This proves Part 1.

It is immediate from the definition of $\boldsymbol{F}$ that the solution $\boldsymbol{s}_t$ with initial condition $\boldsymbol{s}(0) \geq 0$ is an equilibrium solution if and only if for $i = 1, \ldots, m-1$, either $\eta_i(\boldsymbol{s}_0) = 0$ or $s_i = 0$, so the set of equilibrium points equals the set $\cup \mathcal{S}_M$.

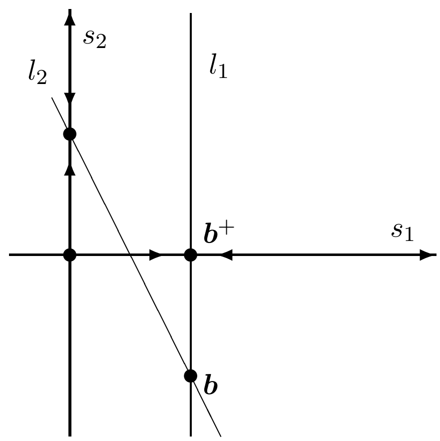
We need to show compactness of the set $\mathcal{S}$. Clearly it is closed. We will show that $\mathcal{S}_M$ has compact closure for all subsets M of $\{1, \ldots, m-1\}$. Consider the restriction of the flow to the set

$$A_M = \{\boldsymbol{s} = (s_i) : s_i = 0 \text{ for } i \in M,\ s_i > 0 \text{ for } i \notin M\}.$$

The fixed points in this satisfy $s_i = 0$ for $i \in M$, and for $\eta_i(\boldsymbol{s}_t) = 0$ for $i \notin M$. After renaming $s_i, i \notin M$, as $t_1, \ldots, t_k$ for some k, these can be written as $PU'\boldsymbol{t} = \boldsymbol{0}$ for the $k \times k$ Gram matrix that is a minor of U obtained by crossing out rows and columns with indices in M. By Proposition 3.5, solutions satisfy $U\boldsymbol{t} = -2\beta'[1]$ for some $\beta < 0$. By Theorem 2 of [Pay], the set of such $\boldsymbol{t}$ in A_M is compact.

Suppose that $\boldsymbol{b} = (b_1, \ldots, b_{m-1})$ is in the boundary of $(\boldsymbol{s} > 0)$ and is in $\mathcal{S}^-$. Then there exists an i so that $b_i = 0$ and $\eta_i(\boldsymbol{b}) < 0$. Because $s_i' = -s_i \eta_i(\boldsymbol{s})$, there exists $\epsilon > 0$ so that $s_i' > 0$ for all points in the open subset $V = \{\boldsymbol{s} > 0 : \|\boldsymbol{s} - \boldsymbol{b}\| < \epsilon\}$ of $(\boldsymbol{s} > 0)$. If $\boldsymbol{s}$ is in V, then $0 < s_i < \epsilon$ and s_i' is positive and bounded below, so $\boldsymbol{s}$ leaves V in finite time. Therefore, $\boldsymbol{b}$ repels all nearby points in $(\boldsymbol{s} > 0)$. $\square$

Figure 2: The flow $\boldsymbol{s}(t)$ for Example 3.12



The next example illustrates the phase portrait for the projectivized Lie bracket flow when there is no positive solution $\boldsymbol{v}$ to $U\boldsymbol{v} = [1]$. To our knowledge, the smallest Gram matrix U' arising from a metric nilpotent Lie algebra is 8×8 (see Example 5.2). For the sake of simplicity, in our example we have chosen a 3×3 matrix U that does not necessarily come from a nilpotent Lie algebra, but for which the $\boldsymbol{a}_t$ flow defined by U and Equation (12) has the same qualitative features as for the $\boldsymbol{a}_t$ flow defined by U'.

Example 3.12. Suppose that

$$U = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 2 \\ 0 & 2 & 3 \end{bmatrix}. \qquad \text{Then} \qquad PU = \begin{bmatrix} 3 & 0 & -3 \\ 2 & 1 & -1 \end{bmatrix},$$

the vectors $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ are $(3, 0, -3)$ and $(2, 1, -1)$, respectively, and $\ker PU$ is spanned by $\boldsymbol{v} = (1, -1, 1)$. This vector $\boldsymbol{v}$ is the solution to $U\boldsymbol{v} = [1]_{3\times 1}$. The hyperplanes $\mathcal{H}_1^0$ and $\mathcal{H}_2^0$ are given by $(a_1 = a_3)$ and $(2a_1 + a_2 - a_3 = 0)$ respectively, and the functions η_1 and η_2 are given by

$$\eta_1\left((s_1, s_2)\right) = 3s_1 - 3$$
$$\eta_2\left((s_1, s_2)\right) = 2s_1 + s_2 - s_3.$$

The hyperplanes intersect in the line $\mathbb{R}\boldsymbol{v}$.

The images $l_1 = [\mathcal{H}_1^0]$ and $l_2 = [\mathcal{H}_2^0]$ in $\mathbb{A}_2$ are the lines $(s_1 = 1)$ and $(2s_1 + s_2 = 1)$ respectively. These lines intersect in the point $\boldsymbol{b} = [\boldsymbol{v}]$ given by $(s_1, s_2) = (1, -1)$.

Let us consider the evolution of $[\boldsymbol{a}_t]$ under time, as measured by (s_1, s_2) coordinates in $\mathbb{A}_2$. It is easy to see that $\eta_1((0,0)) < 0$ and $\eta_2((0,0)) < 0$ as stated in Lemma 3.5. As in Example 3.8, properties of the slope field imply that any point in $(\boldsymbol{s} > 0)$ must approach the fixed point $\boldsymbol{b}^+$, corresponding to the point $(1:0:1)$ in $P^2(\mathbb{R})$.

4 Soliton trajectories for the Ricci flow

4.1 Some solutions of nilsoliton trajectories

Before stating the theorem, we note that all known soliton metrics on nilpotent Lie algebra admit stably Ricci-diagonal bases. It follows from Theorem 2.1 that any time an orthogonal basis $\mathcal{B} = \{X_i\}$ has the property that the sets $\{J_{X_i}\}$ and $\{\mathrm{ad}_{X_i}\}$ are orthogonal, the basis is stably Ricci-diagonal. In many other cases a stably Ricci-diagonal basis is guaranteed to exist:

Proposition 4.1. *Let $(\mathfrak{n}, \mathsf{Q})$ be a soliton metric nilpotent Lie algebra with associated semisimple derivation $D = \mathrm{Ric}_{\mathsf{Q}} - \beta \,\mathrm{Id}$. Suppose that the eigenspaces for D are all one-dimensional, and let $\mathcal{B}$ be a set of orthogonal eigenvectors. The set $\mathcal{B}$ is a stably Ricci-diagonal basis.*

Proof. From [Heb98] it is known that the eigenvalues of D are positive and rational for all $i = 1, \ldots, n$. Write $\mathcal{B} = \{\boldsymbol{x}_i\}_{i=1}^n$ where $D(\boldsymbol{x}_i) = \lambda_i \boldsymbol{x}_i$, with $\lambda_1 < \cdots < \lambda_n$. Because D is a derivation, for all $i < j$, the bracket $[\boldsymbol{x}_i, \boldsymbol{x}_j]$ is in the one-dimensional eigenspace for $\lambda_i + \lambda_j$, so is a multiple of $\boldsymbol{x}_k$ for some $k > j$. Hence the sets $\{J_{\boldsymbol{x}_i}\}$ and $\{\mathrm{ad}_{\boldsymbol{x}_i}\}$ are orthogonal, and by Remark 2.2, the Ricci form is diagonal. □

Theorem 4.2. *Let $(\mathfrak{n}, \mathsf{Q})$ be a metric nilpotent Lie algebra with stably Ricci-diagonal basis $\mathcal{B} = \{\boldsymbol{x}_i\}_{i=1}^n$ with $|\Lambda_{\mathcal{B}}| = m > 0$. Suppose that Q is a soliton inner product with soliton constant β. Let $\mathbf{Ric}_{\mathcal{B}} = (r_1, \ldots, r_n)$ denote the Ricci vector for $(\mathfrak{n}, \mathsf{Q})$ relative to $\mathcal{B}$.*

Let $\mathsf{Q}_t = \sum_{i=1}^n q_i \, dx^i \otimes dx^i$ be the solution to the Ricci flow with initial condition Q. Let $\boldsymbol{a}_t = (a_1(t), \ldots, a_m(t))$ be the structure vector for $(\mathfrak{n}, \mathsf{Q}_t)$ as in Equation (6). Denote $a_1, \ldots, a_m$ also by a_{jk}^l, where (j,k,l) is in $\Lambda_{\mathcal{B}}$.

1. *For $i = 1, \ldots, m$, the function a_i is the solution*
$$a_i(t) = a_i(0)\,(-2\beta t + 1)^{-1}$$
to the differential equation $a_i'(t) = \dfrac{2\beta}{a_i(0)} a_i^2(t)$. The ray $\mathbb{R}\boldsymbol{a}_0$ is invariant under the flow, with
$$\boldsymbol{a}_t = \frac{a_1(t)}{a_1(0)}\, \boldsymbol{a}_0.$$

2. *The Ricci form* $\mathsf{ric}_{\mathsf{Q}_t}$ *for* $(\mathfrak{n}, \mathsf{Q}_t)$ *is diagonal relative to* $\mathcal{B}$ *with diagonal entries given by the Ricci vector for* Q_t *:*

$$\mathbf{Ric}_{\mathcal{B}} = a_1(t)\left(-\frac{1}{2a_1(0)}\,\boldsymbol{a}_0^T Y\right)$$

3. *The solution* $\mathsf{Q}_t = \sum_{i=1}^n q_i\, dx^i \otimes dx_i$ *to the Ricci flow for* $(\mathfrak{n}, \mathsf{Q})$ *is given by*

$$q_j(t) = q_j(0)\,(-2\beta t + 1)^{r_j/\beta},$$

for $j = 1, \ldots, n$.

4. *Let* E_{min} *denote the eigenspace for the minimal eigenvalue of* $\mathsf{ric}_{\mathsf{Q}}$. *The inner product* Q *collapses under the Ricci flow to the equivalence class* $\overline{\mathsf{Q}_\infty}$, *where* Q_∞ *is a semidefinite symmetric bilinear form that is positive definite on* E_{min} *and is degenerate on any subspace properly containing* E_{min}.

Before we prove the theorem, we illustrate it with an example.

Example 4.3. Let $(\mathfrak{n}, \mathsf{Q})$ be the five-dimensional metric Lie algebra that with respect to an orthogonal basis $\mathcal{B} = \{\boldsymbol{x}_i\}_{i=1}^5$ has the following bracket relations:

$$[\boldsymbol{x}_1, \boldsymbol{x}_3] = \boldsymbol{x}_4, \qquad [\boldsymbol{x}_1, \boldsymbol{x}_4] = \boldsymbol{x}_5, \qquad [\boldsymbol{x}_2, \boldsymbol{x}_3] = \boldsymbol{x}_5.$$

By Theorem 2.1, $\mathcal{B}$ is a Ricci diagonal basis. The set $\Lambda_{\mathcal{B}}$ is equal to

$$\{(1,3,4), (1,4,5), (2,3,5)\},$$

and the Gram matrix U and the matrix PU are given by

$$U = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix} \qquad \text{and} \qquad PU = \begin{bmatrix} 2 & -1 & -2 \\ -1 & 2 & -2 \end{bmatrix}.$$

The kernel of PU is spanned by $\boldsymbol{v} = (2,2,1)^T$; note that $U\boldsymbol{v} = 7(1,1,1)^T$. Therefore, by Theorem 2.3, if the inner product Q which has $\mathcal{B}$ as an orthogonal basis has structure vector $\boldsymbol{a} = (\frac{q_4}{q_1 q_3}, \frac{q_5}{q_1 q_4}, \frac{q_5}{q_2 q_3})$ equal to $(2,2,1)^T$, then it is soliton with soliton constant $\beta = -7/2$. In that case the Ricci vector is $-\frac{1}{2}(4,1,3,0,-3)$, and the derivation $D = \mathrm{Ric} - \beta\,\mathrm{Id}$ is represented by $[D]_{\mathcal{B}}$, the diagonal matrix $\mathrm{diag}(3,6,4,7,10)$.

In this example, $a_1 = a_{13}^4, a_2 = a_{14}^5$ and $a_3 = a_{23}^5$. We let $\mathsf{Q} = \sum_{i=1}^5 q_i dx^i \otimes dx^i$ with $q_1 = 1, q_2 = 4, q_3 = 1, q_4 = 2, q_5 = 4$. The soliton metric nilpotent Lie algebra

$(\mathfrak{n}, \mathsf{Q})$ has structure vector $\boldsymbol{a}_0 = (2,2,1)^T$, Ricci vector $-\frac{1}{2}(4,1,3,0,-3)$ and soliton constant $\beta = -7/2$.

By Theorem 4.2, the flow for $\boldsymbol{a}_t$ is given by

$$a_1' = -\tfrac{7}{2}a_1^2, \qquad a_2' = -\tfrac{7}{2}a_2^2, \qquad a_3' = -7a_3^2,$$

solutions of which are

$$a_1(t) = 2\,(7t+1)^{-1}, a_2(t) = 2\,(7t+1)^{-1}, a_3(t) = (7t+1)^{-1}.$$

A solution $\boldsymbol{a}_t$ takes values on the ray $\mathbb{R}^+(2,2,1)$; to be precise,

$$\boldsymbol{a}_t = (7t+1)^{-1}\,(2,2,1).$$

Solutions for the functions $q_1, \ldots, q_5$ are

$$\begin{aligned} q_1(t) &= (7t+1)^{4/7} \\ q_2(t) &= 4\,(7t+1)^{1/7} \\ q_3(t) &= (7t+1)^{3/7} \\ q_4(t) &= 2 \\ q_5(t) &= 4\,(7t+1)^{-3/7}. \end{aligned}$$

Then

$$q_1(t) \asymp t^{4/7}, q_2(t) \asymp t^{1/7}, q_3(t) \asymp t^{3/7}, q_4(t) \asymp t^0, \text{ and } q_5(t) \asymp t^{-3/7},$$

We the notation $f(t) \asymp g(t)$ indicates that for functions $f(t)$ and $g(t)$, the limit $\lim_{t\to\infty} \frac{f(t)}{g(t)}$ is a finite nonzero number.

The inner product collapses to a degenerate volume-normalized symmetric bilinear form $\overline{\mathsf{Q}_\infty}$ supported on $\operatorname{span}\{\boldsymbol{x}_1\}$, which is the eigenspace for the minimal eigenvalue of the semisimple derivation $D = \operatorname{Ric} - \beta \operatorname{Id}$.

Now we prove the theorem.

Proof of Theorem 4.2. Since $(\mathfrak{n}, \mathsf{Q})$ is soliton, by Theorem 2.3, $U\boldsymbol{a}_0 = -2\beta[1]_{m\times 1}$ for some $\beta < 0$. By Theorem 3.1,

$$a_i' = -a_i(U\boldsymbol{a}_t)_i,$$

for $i = 1, \ldots, m$. At the point $\boldsymbol{a} = \lambda\boldsymbol{a}_0$, for any $\lambda > 0$ and any $i = 1, \ldots, m$,

$$\begin{aligned} a_i'(t) &= -a_i(t)\,(U\boldsymbol{a})_i \\ &= -\lambda a_i(t)\,(U\boldsymbol{a}_0)_i \\ &= -\lambda a_i(t)\,(-2\beta[1]_{m\times 1})_i \\ &= 2\beta\lambda a_i(t). \end{aligned} \tag{20}$$

Therefore $\boldsymbol{a}'(t) = 2\beta\lambda\boldsymbol{a}_t$ at all points $\lambda\boldsymbol{a}_0$. Thus, the ray $\mathbb{R}^+\boldsymbol{a}_0$ is invariant under the flow. Since for $i = 1, \dots, m$, the function a_i is given by $a_i(t) = \frac{a_i(0)}{a_1(0)}a_1(t)$ for all $t \geq 0$, $\boldsymbol{a}_t = \frac{a_1(t)}{a_1(0)}\boldsymbol{a}_0$ for all $t \geq 0$.

At the point $\boldsymbol{a}_t = \lambda\boldsymbol{a}_0$, the value of λ is $\frac{a_1(t)}{a_1(0)}$, so Equation (20) becomes

$$a_i'(t) = \frac{2\beta}{a_1(0)}a_1(t)a_i(t) = \frac{2\beta}{a_i(0)}a_i^2(t).$$

Solutions are

$$a_i(t) = a_i(0)\left(-2\beta t + 1\right)^{-1},$$

for $i = 1, \dots, m$. This proves Part 1.

Part 2 is an immediate consequence of Equation (5) in Theorem 2.1.

Now we consider Part 3 of the theorem. In order to compute $q_j(t)$, for $j = 1, \dots, m$, by Theorem 3.1, we need to solve the system of differential equations $(\ln q_j)' = -2(\mathbf{Ric}_{\mathcal{B}})_j$, where $j = 1, \dots, n$. Substituting the expressions for for $a_i(t)$ and $\mathbf{Ric}_{\mathcal{B}}$ from Parts 1 and 2, we have

$$\begin{aligned}(\ln q_j)' &= \sum_{\substack{(j,k,l)\in\Lambda_{\mathcal{B}}\\(k,j,l)\in\Lambda_{\mathcal{B}}}} a_{jk}^l - \sum_{(k,l,j)\in\Lambda_{\mathcal{B}}} a_{kl}^j\\ &= \sum_{\substack{(j,k,l)\in\Lambda_{\mathcal{B}}\\(k,j,l)\in\Lambda_{\mathcal{B}}}} ((a_{kl}^j(0))(-2\beta+1)^{-1} - \sum_{(k,l,j)\in\Lambda_{\mathcal{B}}} ((a_{jk}^l(0))(-2\beta+1)^{-1}.\end{aligned}$$

Integrating, we find that $\ln q_j$ is equal to

$$\sum_{\substack{(j,k,l)\in\Lambda_{\mathcal{B}}\\(k,j,l)\in\Lambda_{\mathcal{B}}}} \frac{a_{jk}^l}{-2\beta}\ln\left(-2\beta t + 1\right) - \sum_{(k,l,j)\in\Lambda_{\mathcal{B}}} \frac{a_{kl}^j}{-2\beta}\ln(-2\beta t + 1) \quad + C.$$

After exponentiating both sides and using that

$$r_j = -\frac{1}{2}\left(\sum_{\substack{(j,k,l)\in\Lambda_{\mathcal{B}}\\(k,j,l)\in\Lambda_{\mathcal{B}}}} a_{jk}^l - \sum_{(k,l,j)\in\Lambda_{\mathcal{B}}} a_{kl}^j\right)$$

by Theorem 2.1, the desired expression for $q_j(t)$ is obtained.

Because inner product Q is soliton, the Lie algebra $\mathfrak{n}$ is orthogonally $\mathbb{N}$-graded by the eigenspaces of the derivation associated to Q. Without loss of generality, let r_1 be the minimal eigenvalue of the Ricci form for Q, and let $E_{\min}$ be the corresponding eigenspace. Then $\lim_{t\to\infty} t^{-r_1/\beta}\mathsf{Q}_t$ is positive definite on $E_{\min}$ but is degenerate on any subspace properly containing $E_{\min}$. This completes the proof of the theorem. $\square$

5 Examples

5.1 Heisenberg metric Lie algebras

The Heisenberg metric Lie algebras are among the most symmetric nilpotent Lie algebras.

Example 5.1. Let Q be an inner product on the $(2r+1)$-dimensional Heisenberg algebra $\mathfrak{h}_{2r+1}$. From Equation (4), it is easy to see that the center $\mathfrak{z}$ is the single positive eigenspace for the Ricci endomorphism, and the Ricci form is negative definite on $\mathfrak{z}^\perp$. A vector $\boldsymbol{z}$ spanning the center is an eigenvector for the Ricci endomorphism. By orthogonally block diagonalizing the nondegenerate skew-symmetric endomorphism $J_{\boldsymbol{z}} : \mathfrak{h}_{2r+1} \to \mathfrak{h}_{2r+1}$ into one block of form $[0]$ and r blocks of form $\left(\begin{smallmatrix} 0 & c \\ -c & 0 \end{smallmatrix}\right)$, it is possible to find an orthogonal Ricci eigenvector basis $\mathcal{B} = \{\boldsymbol{x}_i\}_{i=1}^{2r+1}$ of $\mathfrak{h}_{2r+1}$ such that $\Lambda_{\mathcal{B}} = \{(2i-1, 2i, 2r+1) \,|\, i = 1, \ldots, r\}$. The Gram matrix $U = (u_{ij})$ for $\mathfrak{h}_{2r+1}$ relative to $\mathcal{B}$ is the positive definite $r \times r$ matrix defined by

$$u_{ij} = \begin{cases} 3 & i = j \\ 1 & i \neq j \end{cases}.$$

The solution to $U\boldsymbol{v} = [1]_{r\times 1}$ is a scalar multiple of $\boldsymbol{b} = [1]_{r\times 1}$. Therefore, any inner product $\mathsf{Q}^\star$ with structure vector that is a scalar multiple of $[1]$ is soliton. The $(r-1) \times r$ matrix $PU = (b_{ij})$ is of form

$$b_{ij} = \begin{cases} 2 & i = j \\ 0 & i \neq j, 1 \leq i, j \leq m-1 \\ -2 & j = m \end{cases}$$

and by Remark 3.7, after a change of variables to $\boldsymbol{s}$, the trajectories for the projectivized Ricci flow for $\mathfrak{h}_{2r+1}$ are encoded in the system of differential equations

$$(\ln s_i)' = 2(1 - s_i), \quad i = 1, \ldots, r-1,$$

which has solution

$$s_i(t) = \frac{e^{2t}}{C_i + e^{2t}}, \quad i = 1, \ldots, r-1.$$

Clearly, $\boldsymbol{s}_t$ converges to $[1]$ as $t \to \infty$ for all initial conditions. At the limit point, all values of the structure constants are equal, so that the limiting volume-normalized metric Lie algebra is the Heisenberg Lie algebra endowed with a volume-normalized soliton inner product.

5.2 When the Lie algebra does not support a soliton inner product

Next is an example of a seven-dimensional nilpotent metric Lie algebra $(\mathfrak{n}, \mathsf{Q})$ such that the limit $\overline{(\mathfrak{n}_\infty, \mathsf{Q})}$ of $\overline{(\mathfrak{n}, \mathsf{Q})}$ under the projectivized Ricci flow $\psi_t : \mathcal{N}_7 \to \mathcal{N}_7$ has a limiting Lie algebra $\mathfrak{n}_\infty$ that is not isomorphic to the initial Lie algebra $\mathfrak{n}$.

Example 5.2. Let $\mathfrak{n}$ be a Lie algebra with basis $\mathcal{B} = \{\boldsymbol{x}_i\}_{i=1}^7$ and with algebraic structure determined by the bracket relations

$$\begin{aligned} [\boldsymbol{x}_1, \boldsymbol{x}_i] &= (\alpha_{1,i}^{i+1})\boldsymbol{x}_{i+1} && \text{for } i = 2, \ldots, 6 \text{ and} \\ [\boldsymbol{x}_2, \boldsymbol{x}_i] &= (\alpha_{2,i}^{i+2})\boldsymbol{x}_{i+2} && \text{for } i = 3, 4, 5, \end{aligned}$$

where $\alpha_{1,i}^{i+1} \neq 0$ for $i = 2, \ldots, 6$ and $\alpha_{2,i}^{i+2} \neq 0$ for $i = 2, \ldots, 5$. No Lie algebra of this form admits a soliton inner product (Theorem 34, [Pay]).

Take an initial inner product Q that is diagonal with respect to $\mathcal{B}$. By Theorem 2.1, $\mathcal{B}$ is a Ricci-diagonal basis, and by Remark 2.2 $\mathcal{B}$ remains Ricci-diagonal under rescalings of Q, so it is stably Ricci-diagonal. Let $\boldsymbol{a}_t$ denote the structure vector for the solution Q_t to the Ricci flow at time t. Recall that the entries of a_i are the squares of the nonzero structure constants α_{jk}^l. Since no Lie algebra with $\boldsymbol{a}_t > 0$ can be soliton, the limit

$$\boldsymbol{a}^\star = \lim_{t\to\infty} [\boldsymbol{a}_t] = (a_1 : a_2 : \cdots : a_8)$$

in projective space must have some entry a_i equal to zero.

The only elements $\boldsymbol{x}$ in $\mathfrak{n}$ such that the endomorphism $\mathrm{ad}_{\boldsymbol{x}}$ has rank three or more lie in $\mathrm{span}\{\boldsymbol{x}_1, \boldsymbol{x}_2\}$. For this kind of element $\boldsymbol{x} = c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2$, where $c_1, c_2 \in \mathbb{R}$, the adjoint map is represented with respect to $\mathcal{B}$ by the matrix

$$[\mathrm{ad}_{c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2}]_{\mathcal{B}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c_1\alpha_{1,2}^3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_1\alpha_{1,3}^4 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_2\alpha_{2,3}^5 & c_1\alpha_{1,4}^5 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_2\alpha_{2,4}^6 & c_1\alpha_{1,5}^6 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_2\alpha_{2,5}^7 & c_1\alpha_{1,6}^7 & 0 \end{bmatrix}.$$

In particular, there exist elements $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ such that the ranks of $\mathrm{ad}_{\boldsymbol{x}_1}$ and $\mathrm{ad}_{\boldsymbol{x}_2}$ are five and three respectively.

We claim that if any structure constant becomes zero as $[\boldsymbol{a}_t]$ approaches its limit, the limiting Lie algebra $\mathfrak{n}_\infty$ no longer has this property and therefore is not isomorphic to the original Lie algebra $\mathfrak{n}$. The only way that $\mathfrak{n}_\infty$ can have an element

such that the rank of $\mathrm{ad}_{c_1\boldsymbol{x}_1+c_2\boldsymbol{x}_2}$ is five is if $\alpha_{1,i}^{i+1} \neq 0$ for $i = 2, \ldots, 6$. In order to have an additional element $c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2$ with the rank of $\mathrm{ad}_{c_1\boldsymbol{x}_1+c_2\boldsymbol{x}_2}$ equal to three, it is necessary $\alpha_{2,i}^{i+2} \neq 0$ for $i = 3, 4, 5$. But then $\boldsymbol{a}^\star > 0$, a contradiction. Therefore, $\mathfrak{n}_\infty$ can not be isomorphic to $\mathfrak{n}$.

References

[GIK06] Christine Guenther, James Isenberg, and Dan Knopf, *Linear stability of homogeneous Ricci solitons*, Int. Math. Res. Not. (2006), Art. ID 96253, 30. MR MR2264732

[Ham82] Richard S. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Differential Geom. **17** (1982), no. 2, 255–306. MR MR664497 (84a:53050)

[Heb98] Jens Heber, *Noncompact homogeneous Einstein spaces*, Invent. Math. **133** (1998), no. 2, 279–352. MR 99d:53046

[IJ92] James Isenberg and Martin Jackson, *Ricci flow of locally homogeneous geometries on closed manifolds*, J. Differential Geom. **35** (1992), no. 3, 723–741. MR MR1163457 (93c:58049)

[IJL06] James Isenberg, Martin Jackson, and Peng Lu, *Ricci flow on locally closed homogeneous 4-manifolds*, Communications in Analysis and Geometry **14** (2006), no. 2, 345–386.

[Jab] Michael Jablonski, *Detecting orbits along subvarieties via the moment map*, preprint: arXiv:math.DG/0810.5697.

[Jen69] Gary R. Jensen, *Homogeneous Einstein spaces of dimension four*, J. Differential Geometry **3** (1969), 309–349. MR MR0261487 (41 #6100)

[KM01] Dan Knopf and Kevin McLeod, *Quasi-convergence of model geometries under the Ricci flow*, Comm. Anal. Geom. **9** (2001), no. 4, 879–919. MR MR1868923 (2003j:53106)

[Lau01] Jorge Lauret, *Ricci soliton homogeneous nilmanifolds*, Math. Ann. **319** (2001), no. 4, 715–733. MR 2002k:53083

[Lot07] John Lott, *On the long-time behavior of type-III Ricci flow solutions*, Math. Ann. **339** (2007), no. 3, 627–666. MR MR2336062 (2008i:53093)

[Pay] Tracy L. Payne, *The existence of soliton metrics for nilpotent Lie groups*, `arXiv:0809.5068v1`.

[Pera] Grisha Perelman, *The entropy formula for the Ricci flow and its geometric applications*, `arXiv:mathDG/0211159`.

[Perb] ———, *Finite extinction time for the solutions to the Ricci flow on certain three-manifolds*, `arXiv:mathDG/0307245`.

[Perc] ———, *Ricci flow with surgery on three-manifolds*, `arXiv:mathDG/0303109`.

[Wil82] Edward N. Wilson, *Isometry groups on homogeneous nilmanifolds*, Geom. Dedicata **12** (1982), no. 3, 337–346. MR MR661539 (84a:53048)